Effect of symmetry breaking on two-dimensional random walks

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The distinction between regular and disordered random walks breaks down in two or more spatial dimensions if the regular random walks have broken global spatial symmetries. A better classification for regular random walks is "integrable" and "nonintegrable." It may be impossible to distinguish the dynamics of a nonintegrable regular random walk from the dynamics of a disordered random walk.

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The theory of random walks has had wide application in the physical sciences, ranging from models of transport in solids and molecules to models of biological processes (see Ref. [1] for a review and further references). The treatments generally distinguish random walks on regular lattices from those on disordered lattices. The hope is that on regular lattices it is possible to obtain analytic expressions for return time probabilities, mean first passage times, etc. In this paper, we show that the dynamics underlying random walks is deeply affected by the symmetries of the lattice. A broken spatial symmetry can cause a random walk on a regular lattice to be indistinguishable from that on a disordered lattice.

We will consider a random walker on a two dimensional finite square lattice with $N=(L+1)^2$ lattice sites, such that each site of the lattice has four joining sites (except at the boundaries) and the walker jumps only between nearest neighbor sites. We label the sites (n_1,n_2), where n_1 and n_2 are integers with values $-L/2 \leq n_1 \leq +L/2$ and $-L/2 \leq n_2 \leq +L/2$. The site labeled $\mathbf{n}=(n_1,n_2)$ lies at the spatial point $\mathbf{x}=x_1\hat{\mathbf{i}}+x_2\hat{\mathbf{j}}$, where $x_1=n_1\Delta$ and $x_2=n_2\Delta$, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along the horizontal and vertical axes, respectively, and Δ is the spacing between lattice sites. The probability $P(\mathbf{x},t)$ to find the walker at point $\mathbf{x}$ [site (n_1,n_2)] at time t is described by the master equation

$$\frac{\partial P(\mathbf{x},t)}{\partial t} = \sum_{x_1'=-L\Delta/2}^{L\Delta/2} \sum_{x_2'=-L\Delta/2}^{L\Delta/2} P(\mathbf{x}',t)W(\mathbf{x}'|x) \;, \qquad (1)$$

where the transition matrix $W(\mathbf{x}'|x)=\omega(\mathbf{x}'|x) - \delta_{\mathbf{x},\mathbf{x}'}\sum_{x''}\omega(\mathbf{x}|\mathbf{x}'')$, and $\omega(\mathbf{x}'|\mathbf{x})$ is the transition rate from site $\mathbf{x}'$ to site $\mathbf{x}$. The double summation is over the position of all lattice sites. The walker is constricted to transitions between neighboring sites.

In this paper we will restrict ourselves to systems with detailed balance. Let $P_0(\mathbf{x})=\lim_{t\to\infty}P(\mathbf{x},t)$ denote the long time probability distribution. The random walk satisfies detailed balance if $P_0(\mathbf{x})\omega(\mathbf{x}|\mathbf{x}')=P_0(\mathbf{x}')\omega(\mathbf{x}'|\mathbf{x})$. If this condition is fulfilled, we can symmetrize the transition matrix. Let us introduce a function $\tilde{P}(\mathbf{x},t)=P(\mathbf{x},t)/\sqrt{P_0(\mathbf{x})}$. Then the master equation takes the form $\partial\tilde{P}(\mathbf{x},t)/\partial t=\sum_{\mathbf{x}'}\tilde{P}(\mathbf{x}',t)V(\mathbf{x}'|\mathbf{x})$, where $V(\mathbf{x}'|\mathbf{x})$ is a symmetric matrix defined $V(\mathbf{x}'|\mathbf{x})=\sqrt{P_0(\mathbf{x}')/P_0(x)}[\omega(\mathbf{x}'|\mathbf{x})-\delta_{\mathbf{x},\mathbf{x}'}\sum_{\mathbf{x}''}\omega(\mathbf{x}|\mathbf{x}'')]$. Since $V(\mathbf{x}'|\mathbf{x})$ is symmetric, it has a complete orthonormal set of eigenvectors $|\phi_i\rangle$ and real eigenvalues $\lambda_i (i=0,\ldots,N-1)$. Let $\langle\mathbf{x}|\phi_i\rangle$ denote the entry of the ith eigenvector due to lattice site $\mathbf{x}$. The the solution to the master equation takes the form

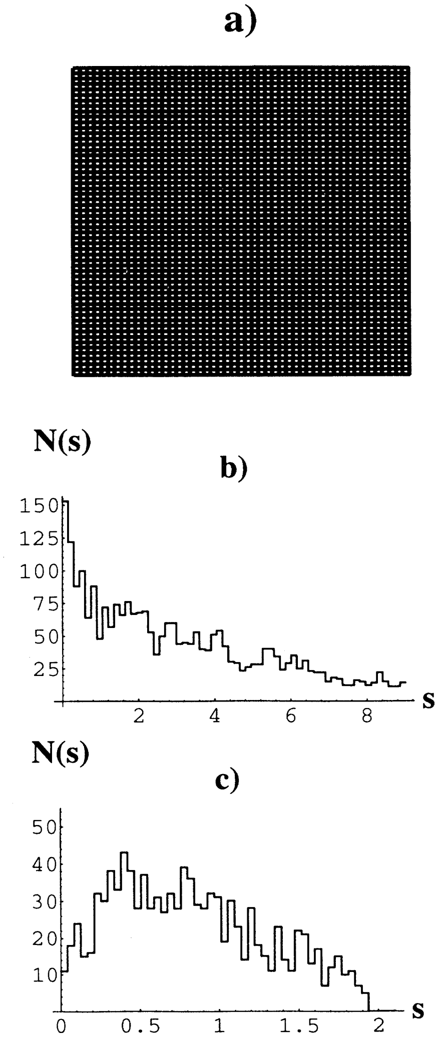

FIG. 1. (a) Square lattice with $L+1=35$ (1225 lattice sites). (b) Nearest neighbor spacing histogram of decay rates for a regular random walk on lattice 1(a). (c) Nearest neighbor spacing histogram of decay rates for a disordered random walk (walk with random transition rates) on lattice 1(a).

$$P(\mathbf{x},t)=\sum_{i=0}^{N-1}\sum_{\mathbf{x}'}\left[\frac{P_0(\mathbf{x})}{P_0(\mathbf{x}')}\right]^{1/2}P(\mathbf{x}',0)\langle \mathbf{x}'|\phi_i\rangle \times e^{\lambda_i t}\langle\phi_i|\mathbf{x}\rangle \ . \tag{2}$$

The eigenvalue $\lambda_0=0$ and eigenvalues $\lambda_i<0$ for $i=1,\ldots,N-1$.

We will now show that because the dynamics of random walks with detailed balance are governed by a symmetric transition matrix, we can use the machinery of quantum chaos theory to categorize them, and relate the behavior of such random walks to an underlying Hamiltonian mechanics. The onset of chaos in classical Hamiltonian systems is due to the breaking of symmetries. For conservative systems with two or more degrees of freedom, symmetry breaking may be caused by symmetry breaking boundaries as occurs in the Bunimovich billiard, or by nonlinear resonances between degrees of freedom of the system such as occur in the Henon-Heiles system [2]. Regardless of the cause, this symmetry breaking and onset of chaos manifests itself in quantum systems as a change in the spectral spacing statistics of the energy-level spectrum. In random walks, as we shall show below, such a symmetry breaking transition manifests itself as a change in the spectral spacing statistics of decay rates.

It is useful first to show the difference between a regular lattice and a disordered lattice. Let us consider a two dimensional flat square lattice [c.f. Fig. 1(a)], where the transition rate from a given lattice site to its nearest neighbors is constant and equal to 1. The sites at the corners of the lattice each have two nearest neighbors, while sites on the edges each have three nearest neighbors. The master equation for this problem is exactly solvable. The decay rates for a lattice with $N=(L+1)^2$ sites are

$$\lambda_{\alpha,\beta}=4-2\cos\left[\frac{\alpha\pi}{L+1}\right]-2\cos\left[\frac{\beta\pi}{L+1}\right], \tag{3}$$

where integers $\alpha=0,..,L$ and $\beta=0,\ldots,L$. A histogram of the nearest-neighbor spacing of decay rates λ is shown in Fig. 1(b) for the case $L+1=35$. It very closely follows a Poisson distribution $P_P=(1/D)\exp(-s/D)$, where s is the spacing and D is the average spacing between levels. The decay rates for this integrable random walk have a large number of close spacings. Let us next consider a random walk on the same lattice [Fig. 1(a)], but now take random values for the transition rates. Random transition rates were obtained from a random number generator (randomly picked from the interval [0,1]). The decay rates and the level spacing statistics for the decay rates

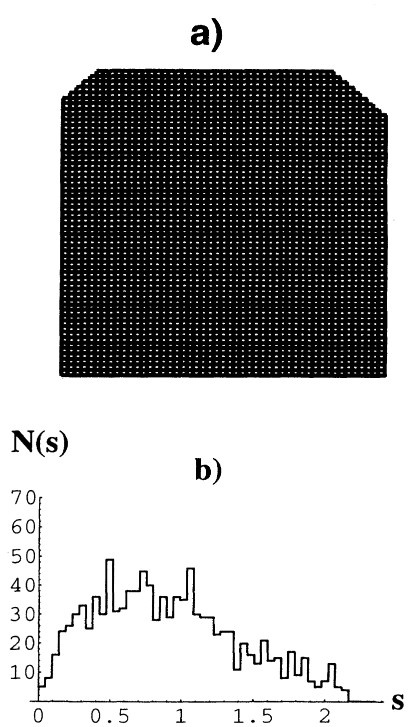

FIG. 2. (a) Square lattice from Fig. 1(a), but with corners cut off. (b) Nearest neighbor spacing histogram of decay rates for a regular random walk on lattice 2(a).

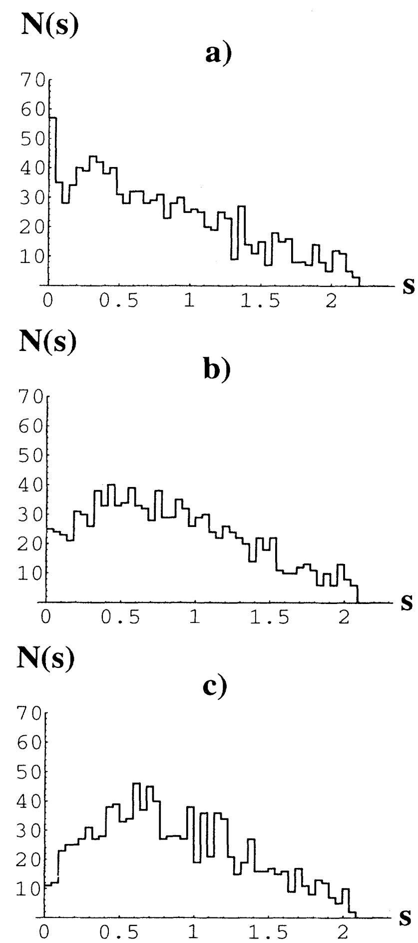

FIG. 3. Nearest neighbor spacing histogram of decay rates for regular random walk on the lattice of Fig. 1(a) with corners cut off (a) by one and two sites, respectively; (b) by two and four sites, respectively; and (c) by three and six sites, respectively.

are computed numerically. The histogram of spacings between neighboring decay rates is shown in Fig. 1(c). The level repulsion is evident and the shape of the distribution, at least numerically, follows closely a Wigner distribution, $P_W(s)=(\pi s/2D^2)\exp(-\pi s^2/4D^2)$, and agrees with predictions by random matrix theory [3].

Let us now break the symmetry of the flat, square random walk by cutting off the corners of the lattice in an uneven way [cf. Fig. 2(a)]. We will again choose the transition rate from a given lattice site to its nearest neighbors to be 1. The histogram of nearest neighbor spacings between decay rates is given in Fig. 2(b). We see that the spectral statistics of this regular random walk is indistinguishable from that of a random walk with random transition rates. We have accomplished this merely by breaking the symmetry of the boundaries on the regular random walk.

However, a question that remains is how sensitive the observed change in spectral statistics is to the relative size of the lattice defect, i.e. the ratio of the size of the defect to the system size. In Fig. 3 we start with one site cut off the top-left corner of the lattice, and two sites cut off the top-right corner. We then increase the amount of the cut proportionally. The result clearly demonstrates that a certain degree of level repulsion sets in immediately at the very small cut. The first case [Fig. 3(a)] demonstrates both Poisson and Wigner-like characters. As the amount of the cut increases, the distribution rapidly changes to Wigner-like.

We can break the symmetry of the lattice in quite a different way. Let us consider a random walker on a square lattice [Fig. 1(a)], but choose the transition rates as functions of the potential

$$U(x_1,x_2)=2x_1^4+\tfrac{3}{5}x_2^4+\epsilon x_1x_2(x_1-x_2)^2 . \qquad (4)$$

The transition rates are

$$\omega(x_1,x_2|x_1\pm\Delta,x_2)=\frac{g}{2\Delta^2}\mp\frac{[U(x_1+\Delta,x_2)-U(x_1,x_2)]}{2\Delta^2}$$

and

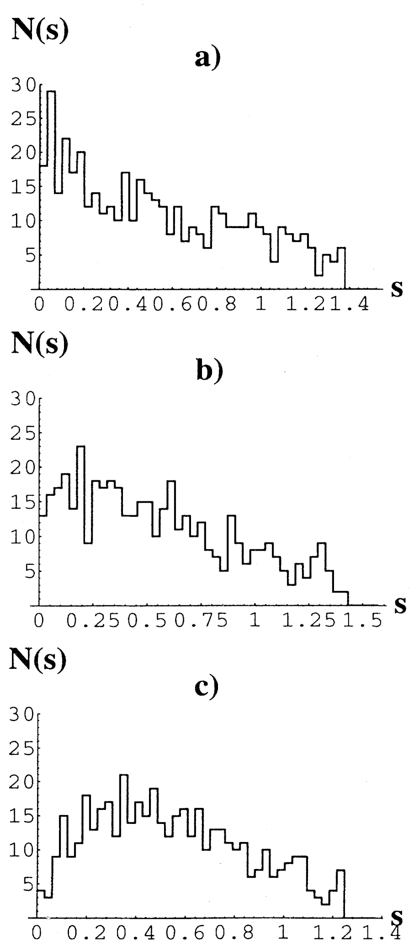

FIG. 4. Nearest neighbor spacing histograms of decay rates for regular random walk on lattice 1(a) with transition rates given by $U(\mathbf{x})$ and (a) $g=0.2$, $\epsilon=0.0$; (b) $g=0.2$, $\epsilon=0.1$; and (c) $g=0.2$, $\epsilon=1.0$.

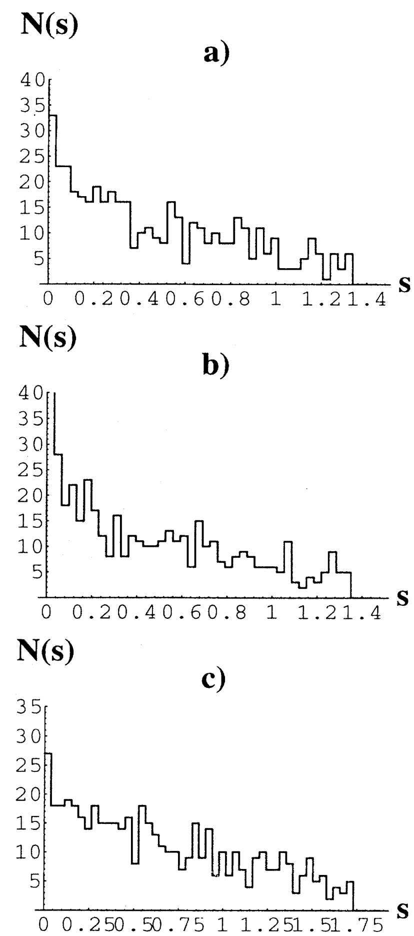

FIG. 5. Nearest neighbor spacing histogram of decay rates for regular random walk on lattice 1(a), with transition rates given by $\tilde{U}(\mathbf{x})$ and (a) $g=0.2$, $\epsilon=0.0$; (b) $g=0.2$, $\epsilon=0.1$; and (c) $g=0.2$, $\epsilon=1.0$.

$$\omega(x_1,x_2|x_1,x_2\pm\Delta) = \frac{g}{2\Delta^2} \mp \frac{[U(x_1,x_2\pm\Delta)-U(x_1,x_2)]}{2\Delta^2} . \quad (5)$$

When $\epsilon=0$, x_1, and x_2 are independent, and the problem decomposes into two independent one-dimensional problems. When $\epsilon\neq 0$, the two degrees of freedom are coupled and can resonate. In Fig. 4, we show the spectral spacing histograms for $\epsilon=0.0$, 0.1, and 1.0. Note the change from a Poisson-like distribution to a Wigner-like distribution as ϵ increases. To support the point that this change is due to coupling and resonance between the degrees of freedom, we will repeat the calculation for a similarly shaped potential $\tilde{U}(x,y)$, where

$$\tilde{U}(x_1,x_2) = -\tfrac{1}{2}\epsilon(x_1-x_2)^2 + \tfrac{13}{80}(x_1-x_2)^4 + \tfrac{13}{80}(x_1+x_2)^4 . \quad (6)$$

Note that there is no coupling of the degrees of freedom for any value of ϵ. The histogram of nearest neighbor spacings of the decay rates for this case is given in Fig. 5 for the same values of ϵ as considered in Fig. 4. The level spacing stays Poisson-like.

It is interesting to look at the limiting case of the walker on the lattice when the lattice spacing goes to zero ($\Delta\to 0$) and the transition rates approach infinity. In this limit, we can make a direct connection to conservative chaos theory. A Kramers-Moyal expansion of the master equation [4,5] leads to a Fokker-Planck equation with corrections of order Δ^2 and smaller,

$$\frac{\partial P(\mathbf{x},t)}{\partial t} = \nabla_{\mathbf{x}}\cdot[P(\mathbf{x},t)\nabla_{\mathbf{x}}U(\mathbf{x})] + \frac{g}{2}\nabla_{\mathbf{x}}^2 P(\mathbf{x},t) + O(\Delta^2) .$$

If small correction terms are neglected, this is exactly the Fokker-Planck equation used by Millonas and Reichl [6] to study Brownian motion in the two dimensional potential $U(\mathbf{x})$. In Ref. [6], the authors showed that the Fokker-Planck equation can be transformed to a Schrodinger-like equation, and a direct connection can be made to classical and quantum chaos theory (a similar connection was made by Alpatov and Reichl [7] for a periodically driven Brownian rotor). The spectral spacing statistics of the decay rates of the Fokker-Planck equation undergo a transition similar to that in Fig. 4(b), and are directly related to a transition to chaos in an underlying Hamiltonian mechanics.

It appears that the distinction between regular and disordered random walks may be much less clear than originally thought. Regular random walks must be categorized as either integrable and nonintegrable depending on whether or not global symmetries have been broken. One can hope to find analytic solutions for integrable regular random walks, but not for nonintegrable regular random walks. One way to check whether a regular random walk is integrable or not is to examine the spectral statistics of its decay rates. The dynamics of nonintegrable regular random walks may be impossible to distinguish from the dynamics of random walks on disordered lattices.

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